Generating function approach to Open Quantum Walks

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Abstract. Open quantum walks (OQWs) have been introduced as a type of quantum walk that is entirely driven by the dissipative interaction with external environments and defined as completely positive trace-preserving maps (CPTP) on graphs. In this work, we study the continuous-time OQW master equation that simulates an OQW on $\mathbb{Z}^+$ in the quantum optical setting. The physical system realizing this OQW is a two-level atom interacting with a quantized mode of electromagnetic radiation at zero temperature in the dispersive regime. We solve this OQW analytically using generating functions. We use the obtained solution for arbitrary initial conditions to explicitly construct the moments of this quantum walk. As an example, the dynamics of the observables (mean, variance) are presented for various parameters.

1. Introduction

Open quantum walks (OQWs) are a class of quantum walks (QWs) which are exclusively based on the non-unitary dynamics induced by the interaction with an environment [1–3]. Mathematically, the non-unitary dynamics are described by the completely positive trace preserving (CPTP) maps [4, 5]. Unlike QWs where the probability of finding the walker results from the quantum interference over the vertices of a graph [6–8], the OQW probability to find the quantum walker on a particular node depends on the structure of the underlying graph, and also on the inner state of the walker. In OQWs, the evolution of the walker is strictly driven by the dissipative interaction with a local environment. As a new framework, it has been suggested that OQWs can be used to perform dissipative quantum computation and to create complex quantum states [9]. The detailed description of the formalism of OQWs can be found in [1–3].

Recently, a quantum optical scheme for the experimental realization of OQWs was proposed [10]. In the proposed quantum optical scheme, a two-level atom plays the role of the "walker" and the Fock states of the cavity mode correspond to the lattice sites of the OQW. Using the small unitary rotations approach [11] the effective dynamics of this system was shown to be an OQW. The presence of spontaneous emission in the system was an essential ingredient for obtaining an OQW. Although this scheme leads to OQW, the dynamics of the walker is not covering all possible behaviors, especially in comparison to the traditional microscopic approaches [12].

The main motivation for performing the research presented in this paper is to solve analytically the OQW master equation developed in [10], use the solution to construct the moments of this quantum walk explicitly and derive the exact solution for the mean and the variance.

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The paper is structured as follows: in Sec. 2 we introduce the model, with reference to [10]; Sec. 3 contains the analytical solution of the OQW master equation and discussions; finally, in Sec. 4 we present the conclusion.

2. Model

As suggest by [10], we consider the following quantum optics set-up: a two-level atom (qubit) interacting with a quantized mode of the electromagnetic radiation at zero temperature. The qubit correspond to the "walker" and the Fock states of the cavity mode with the nodes of the graph. The jumps between different nodes is associated with the action of the annihilation or creation operator on a Fock state. In an experiment setting, there will be processes which lead to dissipative losses. In our model, the only dissipative process considered is the spontaneous emission. The master equation for the system has the following form [4, 13]($\hbar = 1$):

$$\frac{d}{dt}\hat{\rho}(t) = -\imath[\hat{H}_{\text{int}}, \hat{\rho}] + \gamma\mathcal{L}\left[\hat{\sigma}_-, \hat{\sigma}_+\right]\hat{\rho}, \tag{1}$$

where $\mathcal{L}[\hat{X}, \hat{Y}]\hat{\rho} = \hat{X}\hat{\rho}\hat{Y} - (1/2)(\hat{\rho}\hat{Y}\hat{X} + \hat{Y}\hat{X}\hat{\rho})$ is the standard Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) dissipator [14, 15]. The interaction Hamiltonian [16] for the model in the rotation wave approximation (RWA) can be written as

$$\hat{H}_{\text{int}} = \Delta\hat{a}^\dagger\hat{a} + g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-). \tag{2}$$

Here Δ denotes the frequency detuning between the qubit and the field, g is their interaction strength, $\hat{\sigma}_\pm$ are the Pauli raising and the lowering operators for the qubit, satisfying $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$. The operators $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the cavity photons. The constant γ is the coefficient of spontaneous emission. With this experimental setting and moving the system into the dispersive regime ($\frac{g}{\Delta} \ll 1$) one can physically implement an OQW. By applying the small unitary rotations method [11] to the quantum optical master equation (1), then apply the RWA, using $\hat{\rho} = \sum_n \hat{\rho}_n \otimes |n\rangle\langle n|$, one can obtain the following continuous-time OQW master equation (see [10] for the full derivation)

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_n(t) = \imath\frac{g^2}{\Delta}\left[n\hat{\sigma}_z + \frac{\hat{\sigma}_z}{2} + \frac{1}{2}, \hat{\rho}_n\right] + \gamma\left(1 - \frac{2g^2}{\Delta^2}(2n+1)\right)\mathcal{L}\left[\hat{\sigma}_-, \hat{\sigma}_+\right]\hat{\rho}_n \\ + \frac{\gamma g^2}{\Delta^2}\Big((n+1)\,\hat{\sigma}_z\rho_{n+1}\hat{\sigma}_z - n\hat{\rho}_n\Big),\end{aligned} \tag{3}$$

where $|n\rangle$ is the Fock state of the cavity mode and $\hat{\rho}_n$ is the positive operator describing the state of the two-level system. In the next section, we solve this OQW master equation (3) analytically using generating functions.

3. Derivation of moments

In this section we derive the exact expression for the observables, the mean and the variance. The OQW master equation (3) can be written as

$$\begin{aligned}\dot{P}_n(t) &= \frac{\gamma g^2}{\Delta^2}((n+1)P_{n+1} - P_n), \\ \dot{X}_n(t) &= \frac{g^2}{\Delta}(2n+1)Y_n - \frac{A_n}{2}X_n - \frac{\gamma g^2}{\Delta^2}\left((n+1)X_{n+1} + nX_n\right), \\ \dot{Y}_n(t) &= -\frac{g^2}{\Delta}(2n+1)X_n - \frac{A_n}{2}Y_n - \frac{\gamma g^2}{\Delta^2}\left((n+1)Y_{n+1} + nY_n\right), \\ \dot{Z}_n(t) &= -A_n(P_n + Z_n) + \frac{\gamma g^2}{\Delta^2}\left((n+1)Z_{n+1} - nZ_n\right),\end{aligned} \tag{4}$$

where $A_n = \gamma\left(1 - \frac{2g^2}{\Delta^2}(2n+1)\right)$ and the index n runs from 0 to ∞. The functions $P_n, ..., Z_n$ are defined as $P_n(t) = \mathrm{Tr}[\hat{\rho}_n(t)]$ and $K_n(t) = \mathrm{Tr}[\hat{\sigma}_K \hat{\rho}_n(t)]$ i.e., $K_n \in (X_n, Y_n, Z_n)$ ($\hat{\sigma}_K$ is the corresponding Pauli matrix). Using this system of differential equations (4), one can easily obtain a corresponding system of differential equations for P_s, X_s, Y_s and Z_s from the generating function $K_s(x,t) = \sum_{n=0}^{\infty} x^n K_n(t)$,

$$\begin{aligned}
\frac{\partial P_s}{\partial t} &= \frac{\gamma g^2}{\Delta^2}(1-x)\frac{\partial P_s}{\partial x}, \\
\frac{\partial X_s}{\partial t} &= \frac{g^2}{\Delta}Y_s + 2x\frac{g^2}{\Delta}\frac{\partial Y_s}{\partial x} + \gamma\left(\frac{g^2}{\Delta^2} - \frac{1}{2}\right)X_s + \frac{\gamma g^2}{\Delta^2}(x-1)\frac{\partial X_s}{\partial x}, \\
\frac{\partial Y_s}{\partial t} &= -\frac{g^2}{\Delta}X_s - 2x\frac{g^2}{\Delta}\frac{\partial X_s}{\partial x} + \gamma\left(\frac{g^2}{\Delta^2} - \frac{1}{2}\right)Y_s + \frac{\gamma g^2}{\Delta^2}(x-1)\frac{\partial Y_s}{\partial x}, \\
\frac{\partial Z_s}{\partial t} &= 2\gamma\left(\frac{g^2}{\Delta^2} - \frac{1}{2}\right)P_s + 4x\frac{\gamma g^2}{\Delta^2}\frac{\partial P_s}{\partial x} + 2\gamma\left(\frac{g^2}{\Delta^2} - \frac{1}{2}\right)Z_s + \frac{\gamma g^2}{\Delta^2}(3x+1)\frac{\partial Z_s}{\partial x}.
\end{aligned} \tag{5}$$

The first equation for this system at $x = 1$ reduces to $P_s(1,t) = \sum_{n=0}^{\infty}(1)^n P_n(t) = \sum_{n=0}^{\infty} P_n(t)$, where P_n is the probability of finding the walker on the node n, and $\sum_{n=0}^{\infty} P_n(t)$ is the total probability. This implies that $P_s(1,t) = 1$. If we assume that at $t = 0$ the walker is localized at site k ($k \in \mathbb{Z}$) with $\hat{\rho}(0) = \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} \otimes |k\rangle\langle k|$, where $a + b = 1$, $(a,b) \in \mathbb{R}_{\geq 0}$ and $z \in \mathbb{C}$, one can show that $P_s(x,0) = x^k$. Using the method of characteristics and initial condition, one can show that the formal solution for P_s has the form

$$P_s(x,t) = \sum_{n=0}^{k} \binom{k}{n} x^n (1 - e^{-\frac{\gamma g^2}{\Delta^2}t})^{k-n} e^{-\frac{\gamma g^2}{\Delta^2}nt}, \tag{6}$$

where $\binom{k}{n}$ denotes the binomial coefficient and the expression for P_n reads

$$P_n(t) = \begin{cases} \binom{k}{n}(1 - e^{-\frac{\gamma g^2}{\Delta^2}t})^{k-n} e^{-\frac{\gamma g^2}{\Delta^2}nt}, k \geq n \\ 0, \qquad \text{otherwise.} \end{cases} \tag{7}$$

The probability to find the walker at site n (7) is shown in figure 1 for various parameters. Following the same steps with initial conditions $X_s(x,0) = 2x^k$ and $Y_s(x,0) = 2x^k$, we derive and solve for the real part X_n and the imaginary part Y_n of the population coherences

$$\begin{aligned}
X_n(t) &= 2e^{\beta t}\binom{k}{n}\,\mathrm{Re}\left\{\bar{z}e^{-(\imath\delta + k\lambda)t}\left[\frac{\alpha}{\lambda}(1 - e^{\lambda t})\right]^{k-n}\right\}, \\
Y_n(t) &= 2e^{\beta t}\binom{k}{n}\,\mathrm{Im}\left\{\imath z e^{(\imath\delta - k\bar{\lambda})t}\left[\frac{\alpha}{\bar{\lambda}}(1 - e^{\bar{\lambda} t})\right]^{k-n}\right\}.
\end{aligned} \tag{8}$$

Here $\lambda = 2\imath\delta + \alpha$, $\alpha = \frac{\gamma g^2}{\Delta^2}$, $\delta = \frac{g^2}{\Delta}$ and $\beta = 2\gamma(\frac{g^2}{\Delta^2} - \frac{1}{2})$ ($(g, \Delta, \gamma) \in \mathbb{R}_{\geq 0}$). The term $\bar{\eta}$ represent the complex conjugate i.e., $\eta \in (\lambda, z)$. The behavior of equation (8) is shown in figure 2 for various parameters. Using the explicit solution of the function P_s (6) and the initial condition $Z_s(x,0) = x^k(a-b)$, one can derive the solution for the function Z_s

$$Z_s(x,t) = \beta\sum_{m=0}^{k} x^m f(m,t) + 4\alpha k\sum_{m=1}^{k} x^m g(m-1,t) + x^k(a-b)e^{\beta t}, \tag{9}$$

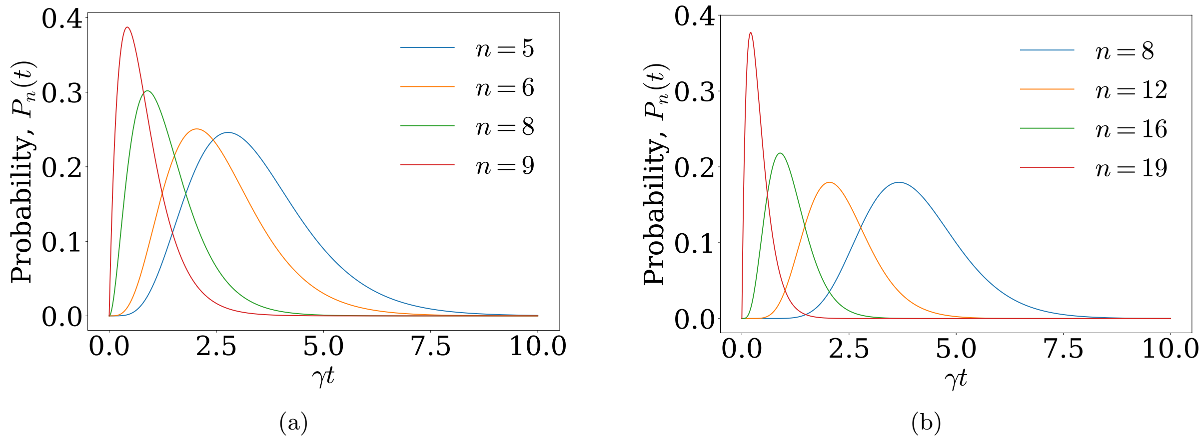

Figure 1: (Color online) The probability $P_n(t)$ to find a walker at site n as a function of dimensionless time γt for different initial Fock states (stated in the legend). The initial sites are $k = 10$ (a) and $k = 20$ (b). Other parameters are set as $\gamma = 0.1$, $g = 0.5$ and $\Delta = 1$.

where

$$f(m,t) = \sum_{n=m}^{k} \binom{k}{n}\binom{n}{m}(-1)^{n-m}\frac{e^{\beta t}}{r_n}\{e^{r_n t}-1\},$$
$$g(m-1,t) = \sum_{n=m}^{k} \binom{k-1}{n}\binom{n}{m-1}(-1)^{n-m+1}\frac{e^{\beta t}}{r'_n}\{e^{r'_n t}-1\}. \quad (10)$$

Here $r_n = -\alpha n - \beta$ and $r'_n = -\alpha(n+1) - \beta$. Other parameters are the same as defined earlier. Using the generating function Z_s (9) one can derive the solution for the population inversion $Z_n(t)$. We are going consider three solutions for Z_s; (i) the population in the vacuum Fock state $m = 0$, (ii) the population in the intermediate site $1 \le m \le k-1$ and (iii) the population in the initial site $m = k$. The solution for the vacuum Fock state $m = 0$ has the form

$$Z_0(t) = \beta \sum_{n=0}^{k} \binom{k}{n}(-1)^n \frac{e^{\beta t}}{r_n}\{e^{r_n t}-1\}. \quad (11)$$

The population in the intermediate site $1 \le m \le k-1$ has the form

$$Z_{1\le m\le k-1}(t) = \beta f(m,t) + 4\alpha k g(m-1,t), \quad (12)$$

where $f(m,t)$ and $g(m-1,t)$ are given by equation (10). Lastly, one can show that the population in the initial site $m = k$ has the form

$$Z_k(t) = \beta f(k,t) + 4\alpha k g(k-1,t) + (a-b)e^{\beta t}$$
$$= \frac{e^{\beta t}}{r_k}(e^{r_k t}-1)(\beta + 4\alpha k) + (a-b)e^{\beta t}, \quad (13)$$

where $r_k = -\alpha k - \beta$. The expressions (11), (12) and (13) conclude the derivation of the solution of $Z_n(t)$. These results are illustrated in figure 3 for various parameters.

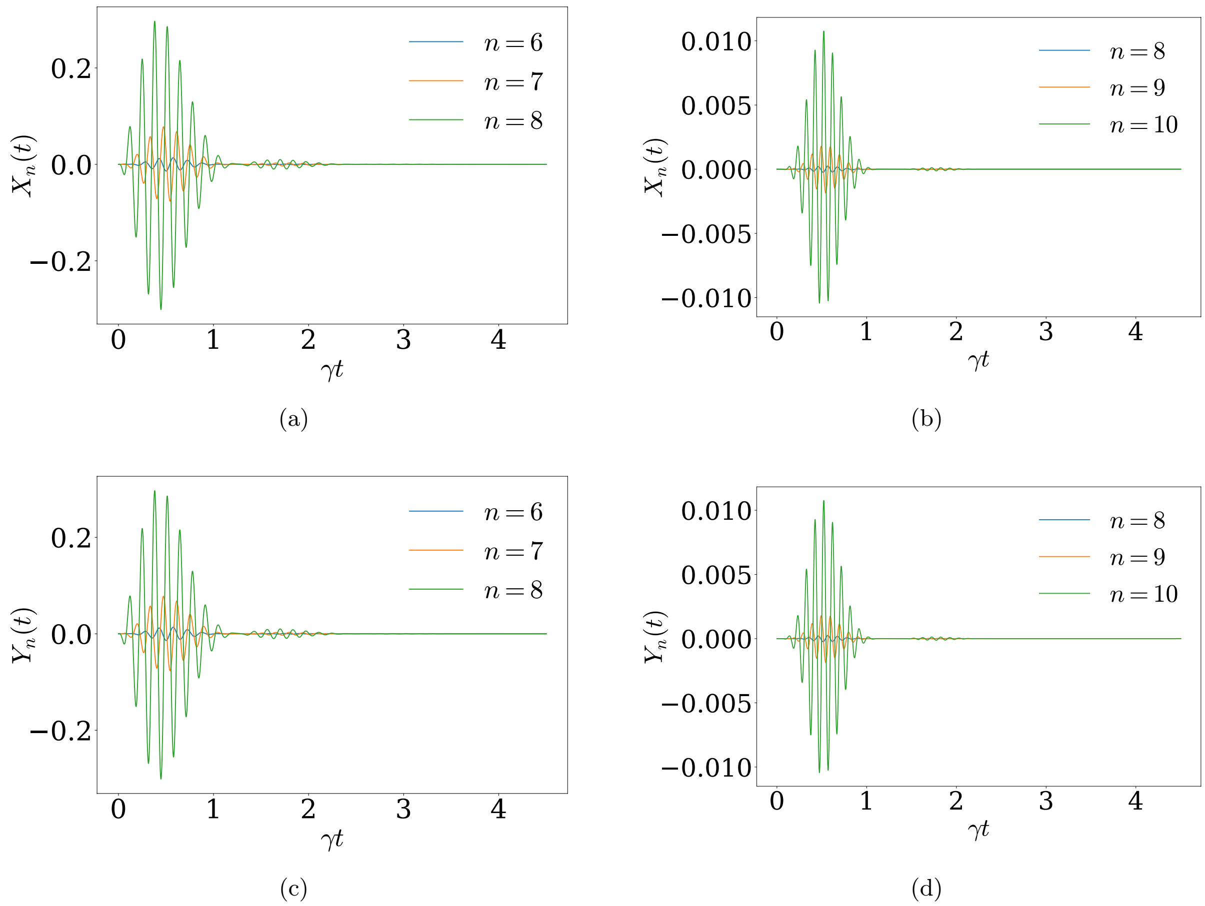

Figure 2: (Color online) The real part $X_n(t)$ (a-b) and the imaginary part $Y_n(t)$ (c-d) of the coherences are shown as a function of the dimensionless time γt for different initial Fock states (stated in the legend). The initial sites are $k = 10$ (a-c) and $k = 15$ (b-d). Other parameters are $\gamma = 0.1$, $g = 0.5$ and $\Delta = 1$.

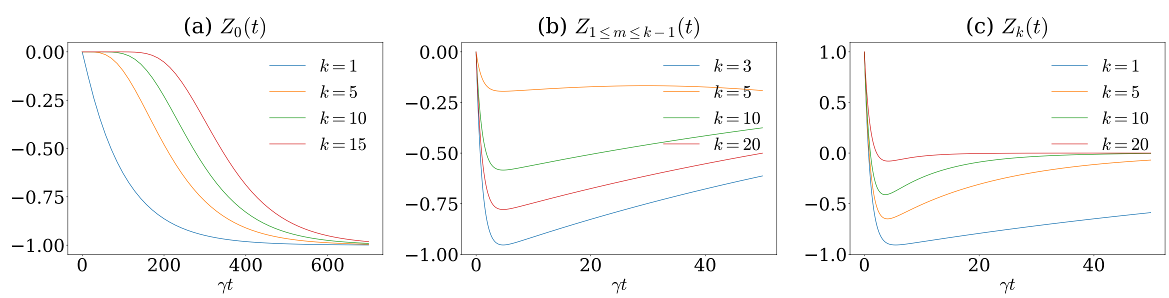

Figure 3: (Color online) The population inversion $Z_n(t)$ is shown as a function of dimensionless time γt for different initial sites k (stated in the legend). The figures correspond to (a) the population in the vacuum Fock state, (b) the population in the intermediate site and (c) the population in the initial site, respectively. Other parameters are $\gamma = 0.1$, $g = 0.1$ and $\Delta = 1$.

In the next step, we are going to derive the exact expression for the mean $\mu(t) = \langle P \rangle$ and the variance $\sigma^2(t) = \langle\langle P \rangle\rangle - \langle P \rangle^2$ of the position of the "walker". Using the explicit solution for $P_n(t)$ (6), the expression for the variables $\langle P \rangle$ and σ^2 reads

$$\begin{aligned} \langle P \rangle(t) &= \sum_{n=0}^{k} n P_n(t,\xi)|_{\xi=1} & \sigma^2(t) &= \sum_{n=0}^{k} \left(n^2 P_n(t,\xi)|_{\xi=1} - \mu^2(t) \right) \\ &= k e^{-\alpha t} & &= k e^{-\alpha t} \left(1 - e^{-\alpha t} \right). \end{aligned} \tag{14}$$

It is clear that at asymptotic time $t \to \infty$ $\langle P \rangle, \langle\langle P \rangle\rangle \to 0$ due to the collapse to the vacuum state. With the help of equation (14), one can derive the velocity distribution V_μ and the velocity spread V_σ^2 (see equation (15)). In the transient regime, for sufficiently large times $t \gg 1$, but still satisfying $|\alpha t| < 1$, the position of the walker obey the central limit theorem with parameters given by

$$V_\mu = \frac{\langle P \rangle(t)}{t} \approx -k\alpha, \qquad V_\sigma^2 = \frac{\sigma^2(t)}{t} \approx k\alpha. \tag{15}$$

4. Conclusion

In this contribution, we solved analytically the OQW master equation derived in [10] using generating functions. We use the obtained solution for arbitrary initial condition to construct the moments of this quantum walk explicitly. The exact solution allowed us to analyze the behavior of the observables of interest for various parameters. An interesting quantum feature (collapse-revival) was observed on the scaled time evolution of the real part and imaginary part of the coherences (see figure 2). This quantum feature provides direct evidence of the field quantization in the cavity. Furthermore, with the help of the analytical solution for the probability $P_n(t)$, the exact solution for the mean and the variance are derived and investigated for transient time.

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References

[1] S. Attal, F. Petruccione, C. Sabot, and I. Sinayskiy, 2012 *J. Stat. Phys.* **147**, 832.
[2] S. Attal, F. Petruccione, and I. Sinayskiy, 2012 *Phys. Lett.* A **376**, 1545.
[3] I. Sinayskiy and F. Petruccione, 2012 *Phys. Scr.* **T 151**, 014077.
[4] H. Breuer and F. Petruccione, 2002 *The Theory of Open Quantum Systems* (Oxford University Press, Oxford).
[5] K. Kraus, 1983 *States, Effects and Operations: Fundamental Notions of Quantum Theory* (Springer-Verlag, Berlin).
[6] Y. Aharonov, L. Davidovich, and N. Zagury, 1993 *Phys. Rev.* A **48**, 1687.
[7] J. Kempe, 2003 *Contemp. Phys.* **44**, 307.
[8] S. E. Venegas-Andraca, 2012 *Quant. Inf. Proc.* **11**, 1015.
[9] I. Sinayskiy and F. Petruccione, 2019 *Eur. Phys. J. Spec. Top.* **227**, 1869-1883.
[10] I. Sinayskiy and F. Petruccione, 2014 *Int. J. Quantum Inform.* **12**, 1461010.
[11] A. B. Klimov and S. M. Chumakov, 2009 *A Group-Theoretical Approach to Quantum Optics* (Wiley-VCH, Darmstadt).
[12] I. Sinayskiy and F. Petruccione, 2015 *Phys. Rev.* A **92**, 0321205.
[13] H. J. Carmichael, 2002 *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations* (Springer, Berlin).
[14] G. Lindblad, 1976 *Commun. Math. Phys.* **48**, 119-130.
[15] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, 1976 *J. Math. Phys.* **17**, 821.
[16] E. T. Jaynes and F. W. Cummings, 1963 *Proc. IEEE.* **51**, 89.